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Some Linear Programs Requiring Many Pivots

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Abstract

We give simple examples of linear programs which use many iterations for the simplex algorithm, emphasizing an algebraic point of view.

Some Linear Programs Requiring Many Pivots

In this note we construct some examples of linear programs which are troublesome for the simplex algorithm. We shall consider two rules for choosing entering variables: most negative reduced cost and maximum objective function improvement. The examples constructed here use essentially the same ideas as in [1, 2, 3]. However, we emphasize an algebraic point of view rather than a geometric one. Also, the numbers may be somewhat neater than in previous constructions.

The basic building block for our examples is the program

	x_1	x_2	z_1
Max x_1			
$x_1 - 8x_2 - 2z_1 \leq 1$	0	0	0
	1	0	0
$2x_1 + 8x_2 - 2z_1 \leq 28$	9	1	0
	11	1	1
$x_1 - 6x_2 + z_1 \leq 42$	27	0	13
	28	0	14
$x_2 \leq 1$	29	1	19

The simplex algorithm with most negative reduced cost rule for entering variable produces the sequence of six iterations above. Note that the variable x_2 starts at zero, goes to its upper limit, returns to zero, then back to its upper limit. To construct harder problems we replace the constraint $x_2 \leq 1$ by a more complicated system of constraints, so that the process of increasing x_2 from zero to its upper limit requires several steps. The next example is:

$$\begin{aligned}
&\text{Max } x_1 \\
&x_1 - 8x_2 - 2z_1 \le 1 \\
&2x_1 + 8x_2 - 2z_1 \le 700 \\
&x_1 - 6x_2 + z_1 \le 1050 \\
&x_2 - 8x_3 - 2z_2 \le 1 \\
&2x_2 + 8x_3 - 2z_2 \le 28 \\
&x_2 - 6x_3 + z_2 \le 42 \\
&x_3 \le 1
\end{aligned}$$

x_1	x_2	x_3	z_1	z_2
0	0	0	0	0
1	0	0	0	0
		. . .		
233	29	1	0	19
235	29	1	1	19
		. . .		
699	0	0	349	0
700	0	0	350	0
		. . .		
729	29	1	495	19

The constraints governing x_2 are the same as those governing x_1 in our first example—we have added one to the subscripts. Thus when x_2 increases from zero to 29 in the first and third (...) six iterations are required.

The second (...) describes the sequence of iterations during which x_2 decreases from 29 to zero. It is essential that while this happens x_3 and z_2 also return to zero, since otherwise the third (...) would not require a full six iterations. Fortunately, the pivots in the second (...) are precisely those in our first example in reverse order.* Thus

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*This "reverse order" property was crucial to the Klee-Minty and Jeroslow constructions. Here it is not essential because our building block has a "forward, backward, forward" character rather than a "forward, backward" character as in the earlier constructions.

the first pivot in the second (...) has $x_2 = 28$, $x_3 = 0$, $z_2 = 14$, the next has $x_2 = 27$, $x_3 = 0$, $z_1 = 13$ and so forth. Our second example thus uses 21 iterations.

In general the next example is obtained from the preceding one by increasing the number of variables by two and the number of constraints by three. We add one to the subscript of all variables in the preceding problem to obtain all but three of the constraints for our new problem. The remaining constraints are $x_1 - 8x_2 - 2z_1 \leq 1$, $2x_1 + 8x_2 - 2z_1 \leq 24M + 4$, and $x_1 - 6x_2 + z_1 \leq 36M + 6$ where M is the optimal objective function value of the previous problem. The new problem will have optimal solution $x_1 = 25M + 4$, $x_2 = M$, and $z_1 = 17M + 2$.

If the old problem uses k iterations the new problem uses 3k + 3 iterations. The problems constructed in this way have sparse constraint matrices in which all coefficients are ± 1, 2, 6, or 8. However, the right-hand sides do grow exponentially (for further discussion of this issue see [3]). All intermediate feasible solutions are integer.

Finally, we wish to show how this construction can be modified to make the simplex algorithm behave badly when the entering variable producing the largest objective function increase is used. In our first example, this algorithm would go from $x_1 = 1$, $x_2 = 0$, $z_1 = 0$ to $x_1 = 27$, $x_2 = 0$, $z_1 = 13$. To prevent this we add the constraint $z_1 - w_1 \leq .6$. Now the choice will be between (9, 1, 0) and (2.2, 0, .6) so the first will be chosen. Similarly, we prevent the algorithm going from (11, 1, 1) to (29, 1, 19) by adding the constraint $-x_1 + 8x_2 + 2z_1 - v_1 \leq 8.5$. Finally, we add the constraints $-x_1 + 8x_2 + z_1 \leq 1$ and $-x_1 - 9x_2 + 2z_1 \leq .1$

to insure that when x_1 goes from its maximum to zero the other variables do also.

As before, we can construct a sequence of problems. The new problem is obtained from the old one by adding one to the subscript of all the variables in the old problem. Then we add seven new constraints. Three of these are the same as those added for the most-negative-cost example ($x_1 - 8x_2 - 2z_1 \leq 1$, etc.). The others are the four constraints described in the preceding paragraph.

The second problem and its sequence of iterations are shown on the next page. The constraint $-x_2 + 8x_3 + z_2 \leq 1$ does not become tight at any iteration. Its effect appears at the iteration following $x_1 = 535.8$, $x_2 = 10.2$, $x_3 = 1$. Without this constraint the next solution would be (699, 0, 1). The effect of $-x_2 - 9x_3 + 2z_2 \leq .1$ appears at (663.8, 2.2, 0). It is curious that these constraints have a substantial adverse effect on this version of the simplex algorithm. Similar remarks apply to the v_1, w_1 constraints which could be eliminated by inspection.

We hope that some of the techniques developed here will be useful in further study of the simplex algorithm, including the open problems in [3].

Max x_1

$$x_1 - 8x_2 - 2z_1 \leq 1$$
$$2x_1 + 8x_2 - 2z_1 \leq 700$$
$$x_1 - 6x_2 + z_1 \leq 1050$$
$$z_1 - w_1 \leq .6$$
$$-x_1 + 8x_2 + 2z_1 - v_1 \leq 8.5$$
$$-x_1 + 8x_2 + z_1 \leq 1$$
$$-x_1 - 9x_2 + 2z_1 \leq .1$$
$$x_2 - 8x_3 - 2z_2 \leq 1$$
$$2x_2 + 8x_3 - 2z_2 \leq 28$$
$$x_2 - 6x_3 + z_2 \leq 42$$
$$x_3 \leq 1$$
$$z_2 - w_2 \leq .6$$
$$-x_2 + 8x_3 + 2z_2 - v_2 \leq 8.5$$
$$-x_2 + 8x_3 + z_2 \leq 1$$
$$-x_2 - 9x_3 + 2z_2 \leq .1$$

x_1	x_2	x_3	z_1	z_2	w_1	w_2	v_1	v_2
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
9	1	0	0	0	0	0	0	0
73	9	1	0	0	0	0	0	0
82.6	10.2	1	0	.6	0	0	0	0
89	11	1	0	1	0	.4	0	0
217	27	0	0	13	0	12.4	0	0
225	28	0	0	14	0	13.4	0	0
229	28.5	.5	0	16.5	0	15.9	0	0
233	29	1	0	19	0	18.4	0	8.5
234.2	29	1	.6	19	0	18.4	0	8.5
235	29	1	1	19	.4	18.4	0	8.5
371	20.5	1	103	10.5	102.4	9.9	0	0
523	11	1	217	1	216.4	.4	0	0
535.8	10.2	1	226.6	.6	226	0	0	0
663.8	2.2	0	322.6	.6	322	0	0	0
683	1	0	337	0	336.4	0	0	0
699	0	0	349	0	348.4	0	0	0
700	0	0	350	0	349.4	0	0	0
700.5	.5	0	352.5	0	351.9	0	0	0
701	1	0	355	0	354.4	0	8.5	0
			(seven more iterations)					
729	29	1	495	19	494.4	18.4	484.5	8.5

References

1. R. Jeroslow, "The Simplex Alogrithm with the Pivot Rule of Maximizing Criterion Improvement," Discrete Mathematics 4, pp. 367-377.

2. V. Klee & G. Minty, "How Good is the Simplex Algorithm?," in Inequalities III.

3. N. Zadeh, "What is the Worst-Case Behavior of the Simplex Algorithm?," Stanford OR Department Technical Report.

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